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Angularly localized Skyrmions

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Abstract

Quantized Skyrmions with baryon numbers $B = 1, 2$ and 4 are considered and angularly localized wavefunctions for them are found. By combining a few low angular momentum states, one can construct a quantum state whose spatial density is close to that of the classical Skyrmion and has the same symmetries. For the $B = 1$ case, we find the best localized wavefunction among linear combinations of $j = \frac{1}{2}$ and $j = \frac{3}{2}$ angular momentum states. For $B = 2$, we find that the $j = 1$ ground state has toroidal symmetry and a somewhat reduced localization compared to the classical solution. For $B = 4$, where the classical Skyrmion has cubic symmetry, we construct cubically symmetric quantum states by combining the $j = 0$ ground state with the lowest rotationally excited $j = 4$ state. We use the rational map approximation to compare the classical and quantum baryon densities in the $B = 2$ and $B = 4$ cases.

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1. Introduction

The connection between the quantum and classical descriptions of a many-body system is an important but rather tricky one. In nuclei, the existence of a rotational band suggests the existence of a static intrinsic classical shape to the nucleus which is not spherically symmetric [1]. It is not obvious how this classical shape arises from first principles, and it is hard to predict the shape, but one can partially reconstruct it from the spectrum.

For a rigid body, the quantum states of various angular momenta are given by Wigner functions $D^j_{sm}(\alpha, \beta, \gamma)$, where α, β, γ are the Euler angles parametrizing the orientation, j is the total angular momentum and s, m are its components with respect to the body-fixed and space-fixed third axis. Symmetries of the body constrain the possible s-values or combinations of s-values that can occur. A classically oriented state is a δ-function in the Euler angles. This can be obtained by taking an infinite linear combination of Wigner functions. For a body with symmetry, one should take a sum of δ-functions on a set of orientations related by symmetry. Even if there is no fundamental rigid body to start with, one can consider these linear combinations. Thus, given a rotational band of states, one can construct a classically oriented state by taking an infinite linear combination of true quantum states of definite angular momentum. The properties of this oriented state (e.g. the particle density) would define the nature of the intrinsic state.

Something like this has been done in certain condensed matter situations. One may construct a classically oriented state when all that is rigorously available is quantum states labelled by angular momentum. Cooper, Wilkin and Gunn [2] have studied a model of rotating states of a Bose condensate trapped in a harmonic well. By numerically combining precise states over a range of angular momenta, a condensate with localized vortices can be obtained, and such vortices may be physically observed.

This localization depends on the system being large. Ideally, the moment of inertia should be almost infinite because in that case, the angular momentum states of different j are almost degenerate, and the angular localization may be achieved at almost no energy cost. Thus, for large nuclei such as Hf^{170}, there are many states in a rotational band, and it is pretty clear that an intrinsic nuclear shape exists [1], whereas for smaller nuclei the picture is less reliable.

An effective field theory treatment of large many-body systems can give angularly (and spatially) localized states much more easily. For example, in the Ginsburg–Landau description of a Bose condensate, classical solutions of the field equation can naturally exhibit the spatial order of an array of vortices. Because of the underlying symmetries, the classical solution is not unique, but is parametrized by collective coordinates describing, say, the centre-of-mass position and angular orientation. To reconstruct quantum states of definite angular momentum, one may quantize the collective coordinates. A critical comparison of exact quantum states and classical solutions of an effective field theory has been carried out for quantum Hall ferromagnets by Abolfath *et al* [3].

In this paper, we shall consider the Skyrme model and its connection with nuclei and their various angular momentum states. The Skyrme model is an effective field theory of pions, with a topological quantum number B that can be identified with baryon number [4]. The modern justification of the model is based on the idea that each nucleon is made of N_c quarks, where $SU(N_c)$ is the gauge group of QCD, so a nucleus of baryon number B is made of a large number, N_cB, of quarks. The Skyrme model has a semi-rigorous standing if N_c is large [5].

The classical Skyrme field equation, like that of the Ginsburg–Landau model, can be solved numerically and much is known about its minimal energy solutions, known as Skyrmions (especially for pion mass equal to zero) [6]. The classical shapes of the solutions, and their symmetries, are known for values of B up to and beyond 20 (and work is underway to take account of the finite pion mass, which has a qualitatively significant effect for $B \gtrsim 10$). These classical shapes could represent the intrinsic shapes of nuclei of modest size.

The shapes obtained have a subtle relation to shapes of nuclei as understood using models based on point nucleons. For example, four-nucleon potential models are used to describe the α-particle, and the classical minimum occurs for a tetrahedral configuration of the nucleons [7]. The Skyrmion with $B = 4$ has cubic symmetry and forms naturally by the merger of four $B = 1$ Skyrmions in an attractive tetrahedral arrangement. The physical α-particle is spherically symmetric, because it has spin zero, and all classical orientations occur with equal probability.

Our aim in this paper is to bridge the gap between the classical Skyrmion shapes and the quantum states of nuclei. The traditional approach has been to quantize the collective coordinates of Skyrmions, seek the lowest energy states consistent with the allowed values of the angular momentum and compare with the ground state properties of nuclei. This approach has some success in reproducing the known spins of nuclei, especially for even baryon number. A table of allowed angular momenta for the ground and first excited states of rotationally quantized Skyrmions has been constructed recently [8]. However, in these quantum states of definite angular momentum, the original Skyrmion shape information is sometimes completely lost.

We cannot consider an infinite linear combination of angular momentum states, as we expect that large angular momenta will lead to Skyrmion deformations, or if these are suppressed, then to infinite energy. Instead, here we take a small combination of low-lying angular momentum states, and partially reconstruct the shape of the classical Skyrmion solution. We shall optimize the angular localization of the Skyrmion within the limited combinations of states at our disposal. Such a finite combination of states has an energy not necessarily very much higher than the ground state. Although these angularly localized states may not be very significant for free nuclei, we believe that they could be useful for understanding small nuclei that are substructures of larger ones.

This paper is restricted to the $B = 1, 2, 4$ Skyrmions. The classical $B = 1$ Skyrmion has spherically symmetric energy and baryon densities. Therefore, we apparently do not have the problem of orientation as the spherical symmetry is preserved after quantization. However, the Skyrmion still has rotational collective coordinates, and we will show that a particular combination of $j = \frac{1}{2}$ and $j = \frac{3}{2}$ states gives a highly localized orientational state. We also show that the ground state of the deuteron (the $j = 1$ quantum state of the $B = 2$ Skyrmion), without an admixture of higher angular momentum states, retains the toroidal symmetry of the classical solution. Forest *et al* have argued that not only the pure deuteron state, but also deuteron clusters within larger nuclei, show the toroidal structure [9]. Finally, we shall show that a combination of $j = 0$ and $j = 4$ collective states of the $B = 4$ cubic Skyrmion gives a state close to the classically oriented Skyrmion.

Section 2 contains a review of the Skyrme model (for more details see [10]), and in section 3 we give an outline of the rational map approximation for Skyrmions, which we use in the later sections to estimate the baryon density of the classical Skyrmions and their quantum states.

2. The Skyrme model

The Skyrme model in geometrical units is defined by the Lagrangian

$$L = \int \left\{ \frac{1}{2} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{16} \mathrm{Tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger]) \right\} \mathrm{d}^3 x, \tag{1}$$

where $U(t, \mathbf{x})$ is an $SU(2)$-valued scalar field. Introducing the $SU(2)$-valued right current $R_\mu = (\partial_\mu U)U^\dagger$, the Lagrangian (1) takes the concise form

$$L = \int \left\{ -\frac{1}{2} \mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16} \mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right\} \mathrm{d}^3 x. \tag{2}$$

The Euler–Lagrange equation which follows from (2) is the Skyrme equation:

$$\partial_\mu \left(R^\mu + \tfrac{1}{4}[R_\nu, [R^\nu, R^\mu]] \right) = 0. \tag{3}$$

Static solutions are the stationary points (either minima or saddle points) of the energy function

$$E = \int \left\{ -\frac{1}{2} \mathrm{Tr}(R_i R_i) - \frac{1}{16} \mathrm{Tr}([R_i, R_j][R_i, R_j]) \right\} \mathrm{d}^3 x. \tag{4}$$

The scalar field U, at fixed time, is a map from $\mathbb{R}^3$ into S^3, the group manifold of $SU(2)$. However, the boundary condition $U \to 1$ implies a one-point compactification of space, so that topologically U: $S^3 \to S^3$, where the domain S^3 is identified with $\mathbb{R}^3 \cup \{\infty\}$. The topological degree of the map U has the explicit representation

$$B = -\frac{1}{24\pi^2}\int \varepsilon_{ijk}\,\mathrm{Tr}(R_i R_j R_k)\,\mathrm{d}^3x. \tag{5}$$

Skyrme identified this integer, B, with baryon number. It is conserved under continuous deformations of the field, including time evolution. The minimal energy static solutions for each B are known as Skyrmions.

3. Rational map ansatz

There is a precise 1–1 correspondence between rational maps and $\mathbb{C}P_1$ lumps in two dimensions and between rational maps and BPS monopoles in three dimensions. If we identify the baryon number B with the monopole number, the energy density distribution of Skyrmions, particularly those with low baryon number, possesses the same symmetries as some specially symmetric monopoles. This strongly suggests that we can use a rational map ansatz to describe Skyrmions [11]. This map is compatible with the topology, symmetry and general structure of Skyrmions. It separates the radial and angular dependance, which proves very convenient, and though not satisfied by the exact Skyrmion solutions, it is a good approximation.

Rational maps are maps from $S^2 \to S^2$, whereas Skyrmions are maps from $\mathbb{R}^3 \to S^3$. One identifies the domain S^2 of the rational map with concentric spheres in $\mathbb{R}^3$ and the target of the rational map S^2 with spheres of latitude on S^3. A point in $\mathbb{R}^3$ can be parametrized by (r, z); r denotes the radial distance and the complex variable z is related via stereographic projection to the usual polar coordinates θ and ϕ by $z = \tan\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}$. A rational map may be written as $R(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials in z. The degree of the rational map, N, is the greater of the algebraic degrees of the polynomials p and q. N is also the topological degree of the map (its homotopy class) as a map from $S^2 \to S^2$.

The value of the rational map R is associated with the unit vector

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}(R+\bar{R}, \mathrm{i}(\bar{R}-R), 1-|R|^2). \tag{6}$$

The ansatz for the Skyrme field, depending on a rational map $R(z)$ and a radial profile function $f(r)$, is

$$U(r, z) = \exp(\mathrm{i}f(r)\hat{\mathbf{n}}_{R(z)}\cdot\boldsymbol{\tau}), \tag{7}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denotes the triplet of Pauli matrices, and $f(r)$ satisfies $f(0) = \pi$, $f(\infty) = 0$.

An $SU(2)$ Möbius transformation of z corresponds to a rotation in physical space; an $SU(2)$ Möbius transformation of R (i.e. on the target S^2) corresponds to an isospin rotation. Both are symmetries of the Skyrme model, and preserve N.

The baryon number for the ansatz (7) is given by

$$B = -\int \frac{f'}{2\pi^2}\left(\frac{\sin f}{r}\right)^2 \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2 \frac{2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} r^2\,\mathrm{d}r. \tag{8}$$

$2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}/(1+|z|^2)^2$ is equivalent to the usual 2-sphere area element $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. The angular part of the integrand,

$$\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2 \frac{2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2}, \tag{9}$$

is precisely the pull-back of the area form $2\mathrm{i}\,\mathrm{d}R\,\mathrm{d}\bar{R}/(1+|R|^2)^2$ on the target sphere of the rational map R, so its integral is 4π times the degree N of the map. Therefore (8) simplifies to

$$B = \frac{-2N}{\pi}\int_0^\infty f'\sin^2 f\,\mathrm{d}r = N. \tag{10}$$

An attractive feature of the rational map ansatz is that it leads to a simple energy expression which can be separately minimized with respect to the rational map R and the profile function f to obtain close approximations to the numerical, exact Skyrmion solutions and having the correct symmetries.

For $B = 1$, the rational map is $R(z) = z$, and this reproduces Skyrme's hedgehog ansatz [4], which is exactly satisfied by the $B = 1$ Skyrmion. For $B = 2$ and $B = 4$, the symmetries of the computed Skyrmions are $D_{\infty h}$ and O_h, respectively, and in each case there is a unique rational map of the desired degree with the given symmetry, which also minimizes the angular part of the energy. They are, respectively,

$$R(z) = z^2, \qquad R(z) = \frac{z^4 + 2\sqrt{3}\mathrm{i}z^2 + 1}{z^4 - 2\sqrt{3}\mathrm{i}z^2 + 1}. \tag{11}$$

In all these cases, we have made a convenient choice of orientation in presenting the maps.

After quantizing the Skyrme field, we will be interested in the behaviour of the wavefunction with respect to different orientations of the Skyrmion configurations. Consequently, all the information we need will be encoded in the angular dependence of the baryon density (9), which only depends on the rational map, and the profile function f will not be of much interest for our purposes.

4. $B = 1$ case

The $B = 1$ Skyrmion is spherically symmetric and takes the hedgehog form

$$U_0(\mathbf{x}) = \exp\{\mathrm{i}f(r)\hat{\mathbf{x}}\cdot\boldsymbol{\tau})\}. \tag{12}$$

If U_0 is the soliton solution, then $U = AU_0A^{-1}$, where A is an arbitrary constant $SU(2)$ matrix, is a static solution as well, and in order to get quantized Skyrmions which are eigenstates of spin and isospin one needs to treat A as a collective coordinate. So substitute

$$U(\mathbf{x}, t) = A(t)U_0(\mathbf{x})A^{-1}(t) \tag{13}$$

in the Lagrangian (1), where $A(t)$ is an arbitrary time-dependent $SU(2)$ matrix. The Lagrangian for A is [12]

$$L = -M + \lambda\,\mathrm{Tr}(\partial_0 A\partial_0 A^{-1}), \tag{14}$$

where M is the Skyrmion mass (static energy) and λ is an inertia constant which may be found numerically.

The $SU(2)$ matrix A can be written as $A = a_0 + \mathrm{i}\mathbf{a}\cdot\boldsymbol{\tau}$, with $a_0^2 + |\mathbf{a}|^2 = 1$, and after the usual quantization procedure one gets the Hamiltonian in terms of a_ξ $(\xi = 0, 1, 2, 3)$:

$$H = M + \frac{1}{8\lambda}\sum_{\xi=0}^{3}\left(-\frac{\partial^2}{\partial a_\xi^2}\right). \tag{15}$$

Because of the constraint $a_0^2 + |\mathbf{a}|^2 = 1$, the wavefunctions $\Psi(A)$ can be expressed as traceless, symmetric and homogeneous polynomials in a_ξ. Ψ gives the amplitude for the various orientations of the Skyrmions, but no matter what Ψ is, the baryon density remains spherically symmetric and equal to the classical baryon density.

Using the isospin and spin operators

$$I_k = \frac{1}{2}\mathrm{i}\left(a_0\frac{\partial}{\partial a_k} - a_k\frac{\partial}{\partial a_0} - \epsilon_{klm}a_l\frac{\partial}{\partial a_m}\right), \qquad J_k = \frac{1}{2}\mathrm{i}\left(a_k\frac{\partial}{\partial a_0} - a_0\frac{\partial}{\partial a_k} - \epsilon_{klm}a_l\frac{\partial}{\partial a_m}\right), \tag{16}$$

Adkins, Nappi and Witten found the normalized wavefunctions for the neutron, proton and Δ-resonance states within the Skyrme model [12]. It is a consequence of the hedgehog structure that the eigenvalues of the total isospin and spin are equal. The wavefunctions we require here have the opposite I_3 and J_3 eigenvalues, and are

$$\begin{aligned}
&\left|n, s_z = \frac{1}{2}\right\rangle = \frac{\mathrm{i}}{\pi}(a_0 + \mathrm{i}a_3), &&\left|p, s_z = -\frac{1}{2}\right\rangle = -\frac{\mathrm{i}}{\pi}(a_0 - \mathrm{i}a_3),\\
&\left|\Delta^-, s_z = \frac{3}{2}\right\rangle = \frac{\sqrt{2}}{\pi}(a_0 + \mathrm{i}a_3)^3, &&\left|\Delta^{++}, s_z = -\frac{3}{2}\right\rangle = \frac{\sqrt{2}}{\pi}(a_0 - \mathrm{i}a_3)^3,\\
&\left|\Delta^0, s_z = \frac{1}{2}\right\rangle = -\frac{\sqrt{2}}{\pi}(a_0 + \mathrm{i}a_3)\left(1 - 3\left(a_1^2 + a_2^2\right)\right),\\
&\left|\Delta^+, s_z = -\frac{1}{2}\right\rangle = -\frac{\sqrt{2}}{\pi}(a_0 - \mathrm{i}a_3)\left(1 - 3\left(a_1^2 + a_2^2\right)\right).
\end{aligned} \tag{17}$$

The best localized wavefunction will be a superposition of these 'pure' $j = \frac{1}{2}$ and $j = \frac{3}{2}$ states.

If we could take into account an infinite number of angular momentum states, the most localized wavefunction would be the Dirac delta function, which may be expressed in the following form:

$$\delta(\mu) = \sum_j (2j+1)\chi^j(\mu), \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{18}$$

where $\chi^j(\mu)$ is the character of the representation $\mathbf{D}^j$ of dimension $2j+1$ and $a_0 = \cos\mu$. However, this wavefunction does not respect the Finkelstein–Rubinstein (FR) constraints [13], which in the case of the $B = 1$ Skyrmion requires that the wavefunction is antisymmetric under $A \to -A$; thus ensuring that the quantized Skyrmion is a fermion. The sum in (18) must therefore be restricted to half-integer values of j, giving the total $\frac{1}{2}(\delta(\mu) - \delta(\mu - \pi))$. This is exactly localized at the two points $A = 1$ and $A = -1$, where $a_0 = \pm 1$.

The entries of the representation matrices are Wigner functions D^j_{sm}, expressed in terms of $a_0, \ldots, a_3$. That is,

$$\mathbf{D}^j = \begin{pmatrix} D^j_{jj} & \cdots & D^j_{j-j} \\ \vdots & \ddots & \vdots \\ D^j_{-jj} & \cdots & D^j_{-j-j} \end{pmatrix}, \tag{19}$$

and the character χ^j is $\mathrm{Tr}\,\mathbf{D}^j$, which only depends on a_0. The diagonal elements contributing to $\chi^{1/2}$ and $\chi^{3/2}$ are

$$\begin{aligned}
&D^{1/2}_{1/2,1/2} = a_0 + \mathrm{i}a_3, \qquad D^{1/2}_{-1/2,-1/2} = a_0 - \mathrm{i}a_3,\\
&D^{3/2}_{3/2,3/2} = (a_0 + \mathrm{i}a_3)^3, \qquad D^{3/2}_{1/2,1/2} = (a_0 + \mathrm{i}a_3)\left(1 - 3\left(a_1^2 + a_2^2\right)\right),\\
&D^{3/2}_{-1/2,-1/2} = (a_0 - \mathrm{i}a_3)\left(1 - 3\left(a_1^2 + a_2^2\right)\right), \qquad D^{3/2}_{-3/2,-3/2} = (a_0 - \mathrm{i}a_3)^3.
\end{aligned} \tag{20}$$

If we truncate the sum (18) at $j = \frac{3}{2}$, we get the following candidate for a (normalized) wavefunction well localized around $a_0 = \pm 1$:

$$\Psi(a_0, a_1, a_2, a_3) = \frac{8}{\pi}\sqrt{\frac{2}{5}}\left(a_0^3 - \frac{3}{8}a_0\right). \tag{21}$$

In terms of nucleon and Δ-resonance states, this can be written as

$$\frac{1}{\sqrt{5}}\left(|\Delta^-\rangle - |\Delta^0\rangle - |\Delta^+\rangle + |\Delta^{++}\rangle - \frac{\mathrm{i}}{\sqrt{2}}(|n\rangle - |p\rangle)\right), \tag{22}$$

with spins as in (17). A more general wavefunction of this type is

$$\Psi(a_0, a_1, a_2, a_3) = \frac{\sqrt{2}}{\pi}\left(\frac{5}{16} + \kappa + \kappa^2\right)^{-1/2}\left(a_0^3 + \kappa a_0\right). \tag{23}$$

The maximum magnitude of Ψ at $a_0 = \pm 1$ occurs when $\kappa = -\frac{3}{8}$, confirming that this is the best localized wavefunction.

Another measure of how well the wavefunction is localized around $a_0 = \pm 1$ is given by the integral

$$\frac{2}{\pi^2}\left(\frac{5}{16} + \kappa + \kappa^2\right)^{-1}\int_0^\pi a_0^2\left|a_0^3 + \kappa a_0\right|^2 \mathrm{d}\Omega. \tag{24}$$

Here $a_0 = \cos\mu$ and $\mathrm{d}\Omega = 4\pi\sin^2\mu\,\mathrm{d}\mu$. After an easy calculation, we find that this integral is maximal when $\kappa = -\frac{1}{4}$, which is close to the value of κ we got before. One more wavefunction worth considering is

$$\Psi = \frac{4}{\pi}\sqrt{\frac{2}{5}}a_0^3, \tag{25}$$

which is as well localized as that with $\kappa = -\frac{3}{8}$ according to criterion (24) and rather simpler. It is the following combination of nucleon and Δ states:

$$\tfrac{1}{2\sqrt{5}}(|\Delta^-\rangle - |\Delta^0\rangle - |\Delta^+\rangle + |\Delta^{++}\rangle - 2\sqrt{2}\mathrm{i}(|n\rangle - |p\rangle)). \tag{26}$$

These localized states are not physically important for isolated nucleons; however, they could be useful for modelling nucleons in interaction. It appears, for example, that the deuteron is not only formed from a proton and neutron but also contains some Δ-resonances [14, 15]. Therefore, considering a superposition of states with different angular momenta is definitely physically meaningful. In [16], the deuteron was modelled by a quantum bound state of two $B = 1$ Skyrmions in the attractive channel, where the relative orientation of the Skyrmions was chosen to maximize the attraction at a short range. It would be interesting if the states of the individual Skyrmions could be approximated by the combined $j = \frac{1}{2}$ and $j = \frac{3}{2}$ states we have discussed here. The dependence of the force between two Skyrmions on their relative orientation is the classical analogue of the tensor force between nucleons, and it appears to automatically lead to an admixture of a Δ-resonance component to each nucleon.

5. $B = 2$ case

The $B = 2$ Skyrmion has $D_{\infty h}$ symmetry and a toroidal shape [17–19]. It occurs at the minimum of the potential for two $B = 1$ Skyrmions in the attractive channel. We take the symmetry axis to be the third body-fixed axis, and the Skyrmion to be in its standard orientation if this coincides with the third Cartesian axis in space. The quantized Skyrmion's wavefunction is a function of the rotational and isospin collective coordinates. (We ignore the translational collective coordinates and set the momentum to zero.) We have to impose FR constraints, which tell us that the ground state has the quantum numbers $(i, j) = (0, 1)$, where i is the total isospin and j is the total spin. This ground state is a rigid body approximation to the deuteron [20]. Since $i = 0$, there is no dependence on the isospin collective coordinates, and the (normalized) wavefunction is

$$\Psi = \sqrt{\frac{3}{8\pi^2}} D^1_{0m}(\alpha, \beta, \gamma). \tag{27}$$

Here α, β and γ are the rotational Euler angles.

Since we are particularly interested in the spatial orientation, we distinguish states differing in m, the third component of the space-fixed spin. The wavefunction we desire is one with the same symmetry properties as the classical solution, and this is

$$\Psi = \sqrt{\frac{3}{8\pi^2}} D^1_{00}(\alpha, \beta, \gamma) = \sqrt{\frac{3}{8\pi^2}} \cos\beta, \tag{28}$$

which is axially symmetric both on the left- and right-hand side (i.e. with respect to the body-fixed symmetry axis and the x_3-axis in space). Ψ has its maximum magnitude at $\beta = 0$ and $\beta = \pi$, corresponding to the $B = 2$ Skyrmion in its standard orientation, and turned upside down, which is classically indistinguishable after an isospin rotation.

Now, we may calculate the nucleon density in the orientational quantum state (28) and find how quantum effects change the density of the classical configuration. We find the expression for the baryon density distribution $\rho_\Psi(\mathbf{x})$ in the physical space (which is interpreted as nucleon density) by averaging the classical baryon density over orientations weighted with $|\Psi|^2$. The density in the quantum state is therefore

$$\rho_\Psi(\mathbf{x}) = \int \mathcal{B}(D(A)^{-1}\mathbf{x})|\Psi(A)|^2 \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma. \tag{29}$$

Here A stands for the $SU(2)$ matrix parametrized by Euler angles α, β, γ and $D(A)$ for the $SO(3)$ matrix associated with A via

$$D(A)_{ab} = \tfrac{1}{2} \operatorname{Tr}(\tau_a A \tau_b A^\dagger). \tag{30}$$

Recall that the rational map $R(z) = z^2$ gives a good approximation to the $B = 2$ Skyrmion solution. It leads, using (9), to the classical baryon density

$$\mathcal{B}(r, z) = \frac{1}{\pi} \left(\frac{1 + |z|^2}{1 + |z|^4} |z| \right)^2 g(r), \tag{31}$$

where $g(r)$ is a radial function. $g(r)$ is unaffected by the quantum averaging, so we ignore it from now on. In polar coordinates, the angular dependence of $\mathcal{B}$ is given by

$$\mathcal{B} = \frac{1}{\pi} \frac{\left(1 + \tan^2\left(\frac{\theta}{2}\right)\right)^2 \tan^2\left(\frac{\theta}{2}\right)}{\left(1 + \tan^4\left(\frac{\theta}{2}\right)\right)^2}, \tag{32}$$

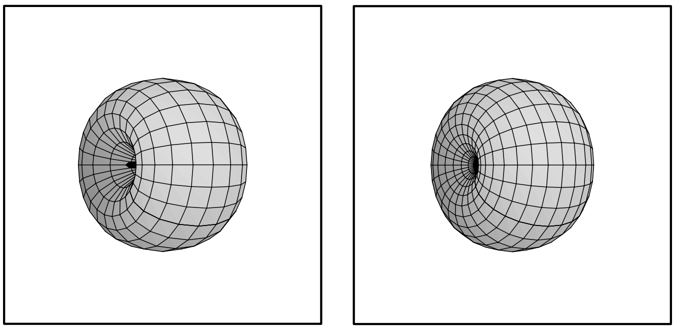
where this is normalized to have angular integral equal to 2, the degree of the rational map.

To evaluate $\rho_\Psi(\mathbf{x})$, we first expand $\mathcal{B}$ in terms of spherical harmonics $Y_{lm}(\theta, \phi)$:

$$\mathcal{B} = \sum_{l,m} c_{lm} Y_{lm}(\theta, \phi), \tag{33}$$

where, because of toroidal symmetry, there are only terms with $m = 0$ and l even. The infinite series (33) is dominated by the first two terms,

$$\mathcal{B} = c_{00} Y_{00}(\theta) + c_{20} Y_{20}(\theta) + \cdots, \tag{34}$$



(*a*) (*b*)

Figure 1. (*a*) Classical baryon density for $B = 2$, equation (34). (*b*) Quantum baryon density for $B = 2$, equation (39).

and all higher terms contribute less than a 5% correction. Because the map R has degree 2, $c_{00} = 1/\sqrt{\pi}$; and we find numerically that $c_{20} = -0.36$. Then $\mathcal{B}(D(A)^{-1}\mathbf{x})$ can be written as

$$\mathcal{B}(\tilde{\mathbf{x}}) = c_{00}Y_{00}(\tilde{\theta}) + c_{20}Y_{20}(\tilde{\theta}), \tag{35}$$

where $\tilde{\mathbf{x}} = D(A)^{-1}\mathbf{x}$ and similarly for $\tilde{\theta}$, $\tilde{\phi}$. Using the transformation properties of spherical harmonics under rotations,

$$Y_{lm}(\tilde{\theta}, \tilde{\phi}) = \sum_k D^l_{mk}(A)^* Y_{lk}(\theta, \phi) \qquad \text{(no sum on } l\text{)}, \tag{36}$$

the fact that $|\Psi|^2 = (3/8\pi^2)D^1_{00}(A)D^1_{00}(A)^*$, the orthogonality properties of the Wigner functions

$$\int D^j_{ab}(A)D^{j'}_{cd}(A)^* \sin\beta \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma = \frac{8\pi^2}{2j+1}\delta^{jj'}\delta_{ac}\delta_{bd}, \tag{37}$$

and finally (in terms of the Wigner $3j$ symbols)

$$\int D^j_{ab}(A)D^{j'}_{cd}(A)D^{j''}_{ef}(A)\sin\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma = 8\pi^2\begin{pmatrix} j & j' & j'' \\ a & c & e\end{pmatrix}\begin{pmatrix} j & j' & j'' \\ b & d & f\end{pmatrix}, \tag{38}$$

we find from (29) that the angular dependence of the nuclear density in the quantum state is

$$\rho_\Psi = c_{00}Y_{00} + \tfrac{2}{5}c_{20}Y_{20}. \tag{39}$$

This is an exact expression—no higher terms are present. We see that it resembles the classical distribution (34), but the first term dominates more. Thus, when quantum effects are included, the classical toroidal density remains, but is smoothed out to become more spherically symmetric. This is illustrated in figures 1(*a*) and (*b*).

6. $B = 4$ case

The $B = 4$ Skyrmion has cubic symmetry; the region of high baryon density resembles a rounded cube with holes in the faces and at the centre [21]. We define the orthogonal body-fixed axes to be those passing through the face centres, and the standard orientation of the cube to be where these axes are aligned with the Cartesian axes in space. We shall again consider the Skyrmion as a rigid body, which means that the configuration is not allowed to vibrate. It was shown in [22] that the ground state, representing the α-particle, has quantum numbers $i = 0$ and $j = 0$, with the (unnormalized) wavefunction $\Psi^{(0)} = 1$ being independent of the rotational and isospin collective coordinates. The first excited state has $i = 0$ and $j = 4$ and is [23]

$$\Psi_m^{(4)} = D^4_{4m}(\alpha, \beta, \gamma) + \sqrt{\frac{14}{5}} D^4_{0m}(\alpha, \beta, \gamma) + D^4_{-4m}(\alpha, \beta, \gamma), \tag{40}$$

this structure being required by the cubic symmetry with respect to body-fixed axes. In [23], the third component of the space-fixed spin, m, was arbitrary.

To make the wavefunction cubically symmetric both on the left- and right-hand side, i.e. also with respect to space-fixed axes, we need to take the linear combination of the above wavefunctions:

$$\Psi^{(4)} = \Psi_4^{(4)} + \sqrt{\frac{14}{5}} \Psi_0^{(4)} + \Psi_{-4}^{(4)}. \tag{41}$$

The cubic symmetry in space is fairly obvious by analogy with (40) and can be verified as follows. First note that symmetry under a 90° rotation about the x^3-axis implies that possible terms in (41) with m other than $\pm 4, 0$ vanish. Now introduce new variables

$$a = \cos\left(\frac{\beta}{2}\right) \mathrm{e}^{\frac{1}{2}\mathrm{i}\gamma}\, \mathrm{e}^{\frac{1}{2}\mathrm{i}\alpha}, \qquad b = -\sin\left(\frac{\beta}{2}\right) \mathrm{e}^{-\frac{1}{2}\mathrm{i}\gamma}\, \mathrm{e}^{\frac{1}{2}\mathrm{i}\alpha}, \tag{42}$$

satisfying $|a|^2 + |b|^2 = 1$. In terms of a and b, the $SU(2)$ orientation matrix parametrized by Euler angles α, β, γ is

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{43}$$

The wavefunctions for different m have the following compact forms:

$$\begin{aligned} \Psi_4^{(4)} &= a^8 + 14a^4b^4 + b^8 \\ \Psi_0^{(4)} &= \sqrt{70}\big(a^4\bar{b}^4 + \bar{a}^4b^4 + \tfrac{1}{40}(3 - 30(|a^2| - |b^2|)^2 + 35(|a^2| - |b^2|)^4)\big) \\ \Psi_{-4}^{(4)} &= \bar{a}^8 + 14\bar{a}^4\bar{b}^4 + \bar{b}^8. \end{aligned} \tag{44}$$

Therefore, the wavefunction (41) in terms of a and b is

$$\begin{aligned} \Psi^{(4)} = {} & 2\,\mathrm{Re}(a^8 + 14a^4b^4 + b^8) + 14(a^4\bar{b}^4 + \bar{a}^4b^4) \\ & + \tfrac{7}{20}(3 - 30(|a^2| - |b^2|)^2 + 35(|a^2| - |b^2|)^4). \end{aligned} \tag{45}$$

As expected, it is real. By acting on A with the generators of the cubic group

$$\begin{pmatrix} \frac{1+\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & \frac{1-\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1+\mathrm{i}}{2} & \frac{1-\mathrm{i}}{2} \\ -\frac{1+\mathrm{i}}{2} & \frac{1-\mathrm{i}}{2} \end{pmatrix}, \tag{46}$$

corresponding to a 90° rotation around a face of the cube, and a 120° rotation around a diagonal of the cube, we find the resulting transformations of (a, b), and it is easy to check that $\Psi^{(4)}$ is cubically symmetric both on the left- and right-hand side.

The wavefunction $\Psi^{(4)}$ has a positive maximum of $\frac{24}{5}$ at the identity, where $(a, b) = (1, 0)$, and at all other elements of the (double cover of the) cubic group. This is as desired, as it corresponds to the Skyrmion having a high probability to be in its standard orientation. But $\Psi^{(4)}$ also has a negative minimum of $-\frac{104}{45}$, which gives a further local maximum of $|\Psi^{(4)}|^2$, at an orientation obtained by a 60° rotation around a diagonal of the cube, which is far from the standard orientation. We wish to suppress this and can do so by adding an arbitrary constant to the wavefunction. This means taking a superposition of the excited state $\Psi^{(4)}$ with the ground state $\Psi^{(0)}$:

$$\Psi = \Psi^{(4)} + \kappa\Psi^{(0)}. \tag{47}$$

Our goal will be to adjust the constant κ to get a nucleon density in the quantum state as close as possible to the classical density.

As in the $B = 2$ case, we define the quantum nuclear density

$$\rho_\Psi(\mathbf{x}) = \int \mathcal{B}(D(A)^{-1}\mathbf{x})|\Psi(A)|^2 \sin\beta \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma, \tag{48}$$

where $\mathcal{B}(\mathbf{x})$ is the classical baryon density of the $B = 4$ Skyrmion in its standard orientation. Using the rational map (11), we find that $\mathcal{B}$ has the angular dependence

$$\mathcal{B} = \frac{12}{\pi}|z|^2(1+|z|^2)^2 \frac{(z^4\bar{z}^4 - z^4 - \bar{z}^4 + 1)}{(z^4\bar{z}^4 + z^4 + 12z^2\bar{z}^2 + \bar{z}^4 + 1)^2}. \tag{49}$$

Expressed in terms of polar angles,

$$\mathcal{B} = \frac{12}{\pi}\tan^2\left(\frac{\theta}{2}\right)\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)^2 \frac{\left(\tan^8\left(\frac{\theta}{2}\right) - 2\tan^4\left(\frac{\theta}{2}\right)\cos 4\phi + 1\right)}{\left(\tan^8\left(\frac{\theta}{2}\right) + 2\tan^4\left(\frac{\theta}{2}\right)\cos 4\phi + 12\tan^4\left(\frac{\theta}{2}\right) + 1\right)^2}, \tag{50}$$

which may be expanded in the following form:

$$\mathcal{B} = d_0 Y_{00} + d_4 Z_4(\theta,\phi) + d_6 Z_6(\theta,\phi) + d_8 Z_8(\theta,\phi) + \cdots. \tag{51}$$

Here Z_4, Z_6 and Z_8 are the unique cubically symmetric combinations of spherical harmonics with, respectively $l = 4$, 6 and 8:[1]

$$\begin{aligned} Z_4 &= Y_{44} + \sqrt{\frac{14}{5}}Y_{40} + Y_{4-4}, \qquad Z_6 = Y_{64} - \sqrt{\frac{2}{7}}Y_{60} + Y_{6-4}, \\ Z_8 &= Y_{88} + \sqrt{\frac{28}{65}}Y_{84} + \sqrt{\frac{198}{65}}Y_{80} + \sqrt{\frac{28}{65}}Y_{8-4} + Y_{8-8}. \end{aligned} \tag{52}$$

The leading coefficient is $d_0 = 2/\sqrt{\pi}$ because the rational map has degree 4, and by numerical calculation we find that $d_4 = -0.28$, $d_6 = -0.032$ and $d_8 = 0.024$. Then, by a similar calculation as in the $B = 2$ case, normalizing the wavefunction, and using the orthogonality properties of the Wigner functions and identity (38), we find the following numerical result for the angular dependence of the quantum baryon density:

$$\rho_\Psi = d_0 Y_{00} + \frac{4}{2.56+\kappa^2}\{-(0.038+0.075\kappa)Z_4 - 0.006Z_6 + 0.002Z_8\}. \tag{53}$$

This is again a finite sum, all the further terms being zero.

The quantum averaging inevitably makes the density in the quantum state (53) closer to spherically symmetric than the classical Skyrmion density (51). Let us now adjust κ so that ρ_Ψ is as close as possible to the classical density, i.e. let us maximize the coefficient

$$\frac{4}{2.56+\kappa^2}(0.038+0.075\kappa) \tag{54}$$

of the $l = 4$ terms. The maximum is at $\kappa \cong 1.17$, which leads to the following expression for the quantum nuclear density:

$$\begin{aligned} \rho_\Psi &\cong 1.13Y_{00} - 0.13Z_4 - 0.006Z_6 + 0.002Z_8 \\ &\cong d_0 Y_{00} + 0.46 d_4 Z_4 + 0.2 d_6 Z_6 + 0.1 d_8 Z_8. \end{aligned} \tag{55}$$

Thus in the $B = 4$ case, as in the $B = 2$ case, one can find a quantum state which localizes the Skyrmion close to its standard orientation, and which preserves the symmetry of the classical solution. However, the inclusion of quantum effects approximately halves the leading non-constant harmonics, here with $l = 4$. This is illustrated in figures 2(*a*) and (*b*).

[1] These can be derived by combining the generating, cubically symmetric Cartesian polynomials $x^2 + y^2 + z^2$, $x^4 + y^4 + z^4$, $x^6 + y^6 + z^6$ and finding the combinations which satisfy Laplace's equation [24].

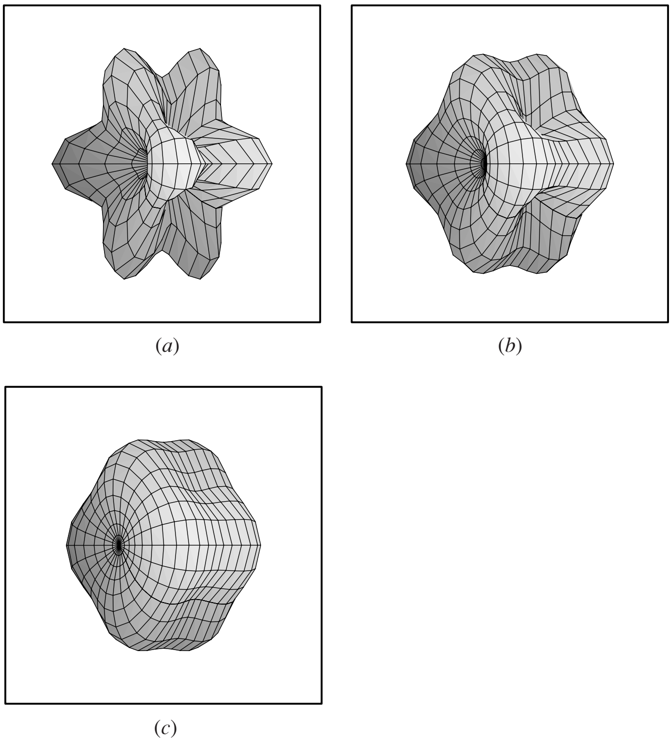


Figure 2. (*a*) Classical baryon density for $B = 4$, equation (51). (*b*) Quantum baryon density for $B = 4$, equation (51). (*c*) Quantum baryon density for $B = 4$ in pure $j = 4$ state, equation (56).

If we considered the pure $j = 4$ state $\Psi^{(4)}$, we would get

$$\rho_{\Psi^{(4)}} \cong d_0 Y_{00} + 0.2 d_4 Z_4 + 0.3 d_6 Z_6 + 0.15 d_8 Z_8, \tag{56}$$

which is much closer to spherically symmetric (see figure 2(*c*)).

We can also find the energy of our state Ψ; it is

$$E = \frac{1}{2.56 + \kappa^2}(2.56 E_{j=4} + \kappa^2 E_{j=0}) \cong 0.65 E_{j=4} + 0.35 E_{j=0}, \tag{57}$$

so it is not as highly excited as a pure $j = 4$ state.

The combination of $j = 0$ and $j = 4$ states, Ψ, is a bit artificial as the quantum state of a free $B = 4$ Skyrmion, but would make sense if we were dealing with interacting $B = 4$ Skyrmions (for example, to describe larger nuclei such as Be^8, C^{12} in the Skyrme model equivalent of the α-particle model). Here we expect the relative orientations of the $B = 4$ subclusters to be rather precisely fixed when they are close together, so as to minimize their potential energy.

7. Conclusions

We have quantized the $B = 1$, $B = 2$ and $B = 4$ Skyrmions being guided by the symmetry properties of the classical solutions, and the corresponding ‘best localized’ wavefunctions were obtained. For the $B = 4$ case, we found a suitable combination of two low-lying angular momentum states which is energetically more efficient than a pure $j = 4$ state and whose density distribution is closer to the classical one. These states may also be considered as an attempt to answer the criticism of the Skyrme model based on the argument that real nuclei do not look like classical Skyrmions. Here we have shown that there exist quantum states of Skyrmions whose density distribution is very similar to the classical one. The states of real nuclei, constructed from the same combination of angular momentum states, will have the same symmetry and may well have a density distribution similar to those of the classical Skyrmions. We have argued that these particular states are those most likely to be realized in certain circumstances. We think that this strongly suggests that, although the found excited states are difficult to observe experimentally, they do exist and in principle might be detected.

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